

Two-point generating function of the free energy for a directed polymer in a random medium

Sylvain Prolhac[‡] and Herbert Spohn[§]

Zentrum Mathematik and Physik Department,
Technische Universität München,
D-85747 Garching, Germany

Abstract. We consider a 1+1 dimensional directed continuum polymer in a Gaussian delta-correlated space-time random potential. For this model the moments (= replica) of the partition function, $Z(x, t)$, can be expressed in terms of the attractive δ -Bose gas on the line. Based on a recent study of the structure of the eigenfunctions, we compute the generating function for $Z(x_1, t)$, $Z(x_2, t)$ under a particular decoupling assumption and thereby extend recent results on the one-point generating function of the free energy to two points. It is established that in the long time limit the fluctuations of the free energy are governed by the two-point distribution of the Airy process, which further supports that the long time behavior of the KPZ equation is the same as derived previously for lattice growth models.

Keywords: Directed polymer, Lieb-Liniger Bose gas, Kardar-Parisi-Zhang equation

PACS numbers: 02.30.Ik 05.20.-y 05.30.Jp 05.40.-a 05.70.Np

[‡] prolhac@ma.tum.de

[§] spohn@ma.tum.de

1. Introduction

Directed polymer in a random medium is a widely studied model in the statistical mechanics of disordered systems [1, 2]. The polymer chain is immersed in a static random potential. In the directed version there is a singled out direction, also referred to as “time”, such that the polymer is constrained to move forward along the time direction. Prominent realizations are domain walls in two dimensional disordered magnets [3], tear lines for paper sheets [4] and vortex lines in disordered superconductors [5]. For spatial dimension $d \leq 2$, the polymer is superdiffusive at any coupling strength, while for $d > 2$, there is a weak coupling phase with diffusive behavior and a still little explored strong coupling phase, see [6, 7] and references therein to earlier work.

Recently there has been considerable progress [8, 9, 10, 11, 12, 13, 14] for a particular continuum version of the directed polymer in $1 + 1$ dimension with both endpoints fixed (point-to-point directed polymer). The free polymer is modeled by a continuum Brownian motion with bending coefficient γ . Then, in the presence of a disorder potential V , the point-to-point partition function of the polymer at inverse temperature β reads

$$Z(x, t) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}[x(\tau)] \exp \left(-\beta \int_0^t d\tau \left[\frac{1}{2} \gamma (\partial_\tau x(\tau))^2 + V(x(\tau), \tau) \right] \right). \quad (1.1)$$

The partition function is the sum over all possible paths $x(\tau)$ of the polymer, starting at position 0 at time 0 and ending at position x at time t . The energy of the polymer is the sum of the elastic bending energy, proportional to γ , and the potential energy obtained from summing the external potential V along the polymer chain. The partition function $Z(x, t)$ is random as inherited from the randomness of the potential V , which is assumed to have a Gaussian distribution with mean 0 and covariance

$$\langle V(x, \tau) V(x', \tau') \rangle = D \delta(x - x') \delta(\tau - \tau'). \quad (1.2)$$

The particular choice of the covariance (1.2) allows to express the n -th moment of Z as a propagator matrix element of an n -particle attractive δ -Bose gas on the line, a model which can be solved exactly by the Bethe ansatz. As will be explained in more detail below, the progress alluded to refers to an exact computation of the generating function for the partition sum $Z(x, t)$. In our contribution, this result will be extended to the generating function jointly of $Z(x_1, t)$ and $Z(x_2, t)$, *i.e.* for two distinct positions x_1 and x_2 of the endpoint of the polymer at the same time t .

An additional interest in the continuum directed polymer comes from the connection to the Kardar-Parisi-Zhang (KPZ) equation [15], which is a stochastic evolution for a growing surface. If we denote the height profile by $h(x, t)$, then, in the conventional units, the one-dimensional KPZ equation reads

$$\partial_t h(x, t) = \frac{1}{2} \lambda (\partial_x h(x, t))^2 + \nu \partial_x^2 h(x, t) + \eta(x, t). \quad (1.3)$$

Here, λ is the strength of the nonlinear growth velocity, ν is the parameter governing the surface relaxation, and η is a white noise with strength \sqrt{D} modeling the random nucleation and deposition events at the surface.

The partition function of the directed polymer $Z(x, t)$ given in Eq. (1.1) satisfies

$$\partial_t Z(x, t) = \frac{1}{2\beta\gamma} \partial_x^2 Z(x, t) - \beta V(x, t) Z(x, t) , \quad (1.4)$$

from which it follows that the free energy, defined by

$$F(x, t) = -\frac{1}{\beta} \log Z(x, t) , \quad (1.5)$$

is a solution of the KPZ equation (1.3) under the identification

$$h = -F , \quad \lambda = \gamma^{-1} , \quad \nu = (2\beta\gamma)^{-1} , \quad \eta = -V . \quad (1.6)$$

In the context of surface growth, the joint distribution of the free energy is a natural quantity of interest: for given time t , it is the joint height statistics at the two spatial reference points x_1 and x_2 , which in particular determines the height-height correlations at time t .

Our considerations are somewhat formal, since taken literally $\langle Z(x, t) \rangle = \infty$. In dimension $d = 1$, this divergence can be easily taken care of by a suitable free energy renormalization, as discussed in detail in [16]. After renormalization one finds that

$$\langle Z(x, t) \rangle = (\beta\gamma/2\pi t)^{-1/2} \exp\left(-\frac{\beta\gamma x^2}{2t}\right) . \quad (1.7)$$

Our paper is organized as follows. In Section 2, we recall the replica method and the mapping to the δ -Bose gas. To provide some background, we explain the one-point generating function for the partition sum and the corresponding Fredholm determinant. The extension to two points is discussed and the long-time limit is obtained. The technical derivation is carried out in Sections 3 and 4 with supporting material in the Appendices.

2. Main results

2.1. Scale invariance, stationarity

To have a short hand the Brownian motion average in (1.1) is denoted by $\mathbb{E}_{(x,t)}^{\beta\gamma}$. To define it we first introduce the Gaussian average $\mathbb{E}^{\beta\gamma}$ with mean 0 and covariance

$$\mathbb{E}^{\beta\gamma}(x(\tau)x(\tau')) = (\beta\gamma)^{-1} \min(\tau, \tau') , \quad (2.1)$$

which corresponds to the Brownian motion starting at 0, and set

$$\mathbb{E}_{(x,t)}^{\beta\gamma}(\cdot) = \mathbb{E}^{\beta\gamma}(\cdot \delta(x(t) - x)) . \quad (2.2)$$

In particular

$$\mathbb{E}_{(x,t)}^{\beta\gamma}(\delta(x(t) - x)) = (\beta\gamma/2\pi t)^{-1/2} \exp\left(-\frac{\beta\gamma x^2}{2t}\right) . \quad (2.3)$$

Let us denote the partition function (1.1) by $Z_{\beta,\gamma,D}(x, t)$ to indicate explicitly the parameter dependence. From the scale invariance of white noise, $V(ax, bt)$ has the

same distribution as $(ab)^{-1/2}V(x, t)$, and of the free directed polymer, $x(at)$ has the same distribution as $a^{1/2}x(t)$, one obtains

$$Z_{\beta, \gamma, D}(x, t) = \beta^3 \gamma D Z_{1,1,1}(\beta^3 \gamma D x, \beta^5 \gamma D^2 t) . \quad (2.4)$$

Thus, it suffices to consider the case when all parameters are equal to one, and from now on we adopt the convention

$$\beta = 1 , \quad \gamma = 1 , \quad D = 1 , \quad Z(x, t) = Z_{1,1,1}(x, t) , \quad \mathbb{E}_{(x,t)}^{\beta\gamma} = \mathbb{E}_{(x,t)} . \quad (2.5)$$

The free energy (1.5) has a systematic upward curvature of $x^2/2t$, compare with (1.7). However the distribution of $F(x, t) - x^2/2t$ is independent of the position x . This fact can be seen by performing a linear change of variables in the functional integration (1.1) defining $Z(x, t)$. One obtains

$$\begin{aligned} \langle Z(x+a, t) \rangle^{-1} Z(x+a, t) &= \langle Z(x, t) \rangle^{-1} \mathbb{E}_{(x,t)} \left(\exp \left(- \int_0^t d\tau V(x(\tau) + a\tau/t, \tau) \right) \right) \\ &= \langle Z(x, t) \rangle^{-1} Z(x, t) , \end{aligned} \quad (2.6)$$

where in the second equality we used that the white noise $V(x, \tau)$ is statistically translation invariant in the spatial argument. Hence the distribution of $Z(x, t)/\langle Z(x, t) \rangle$ is independent of x . More general, the stochastic process $x \mapsto Z(x, t)/\langle Z(x, t) \rangle$ is stationary in x , and correlation functions of the form

$$\langle f_1(Z(x_1, t)/\langle Z(x_1, t) \rangle) \dots f_n(Z(x_n, t)/\langle Z(x_n, t) \rangle) \rangle \quad (2.7)$$

are invariant under a global translation of all the arguments. In particular, the joint distribution of $Z(x_1, t)/\langle Z(x_1, t) \rangle$ and $Z(x_2, t)/\langle Z(x_2, t) \rangle$ depends only on the separation $x_2 - x_1$.

For our computation of the two-point generating function it will be of advantage to keep the dependence on x_1, x_2 separately. The final result will confirm the parabolic free energy shift and the dependence on $x_2 - x_1$ only.

2.2. Replicas

The n -point correlation function of $Z(x, t)$ can be computed by introducing the replicas $x_1(\tau), \dots, x_n(\tau)$, which are simply independent copies of the free directed polymer $x(\tau)$. More specifically, using the explicit form of the generating function for a Gaussian,

$$\begin{aligned} \langle Z(x_1, t) \dots Z(x_n, t) \rangle &= \left\langle \prod_{j=1}^n \mathbb{E}_{(x_j, t)} \left(\exp \left(- \int_0^t d\tau V(x_j(\tau), \tau) \right) \right) \right\rangle \\ &= \left(\prod_{j=1}^n \mathbb{E}_{(x_j, t)} \right) \left(\exp \left(\frac{1}{2} \int_0^t d\tau \sum_{i \neq j=1}^n \delta(x_i(\tau) - x_j(\tau)) \right) \right) , \end{aligned} \quad (2.8)$$

where in the second line the average is over all replicas, the j -th replica starting at 0 and ending at x_j at time t . It should be noted that the literal Gaussian average would also include the self-interaction term $-\frac{1}{2} \sum_{i=1}^n \delta(x_i - x_i)$. Thus, for the moments of Z , the free energy renormalization needed to properly define (1.1) simply corresponds to subtract the self-energy.

The Feynman-Kac formula implies that the n -point correlation function $\langle Z(x_1, t) \dots Z(x_n, t) \rangle$ satisfies the imaginary time Schrödinger equation

$$-\partial_t \langle Z(x_1, t) \dots Z(x_n, t) \rangle = H_n \langle Z(x_1, t) \dots Z(x_n, t) \rangle \quad (2.9)$$

with the initial condition $Z(x_j, 0) = \delta(x_j)$. Here H_n is the Lieb-Liniger quantum Hamiltonian of n particles on the line with an attractive δ -interaction,

$$H_n = -\frac{1}{2} \sum_{i=1}^n (\partial_{x_i})^2 - \frac{1}{2} \sum_{i \neq j=1}^n \delta(x_i - x_j) \quad (2.10)$$

[17, 18]. In this representation the free energy renormalization corresponds to the normal ordering of H_n . “Solving” (2.9), the n -point correlation function is given by

$$\langle Z(x_1, t) \dots Z(x_n, t) \rangle = \langle x_1, \dots, x_n | e^{-tH_n} | 0 \rangle. \quad (2.11)$$

Here $|0\rangle$ is the state where all particles are at 0 and $|x_1, \dots, x_n\rangle$ is the one where the j -th particle is at x_j . Since $|0\rangle$ is symmetric under the exchange of particle labels, one can symmetrize in the final state. Thus the propagator e^{-tH_n} is needed only in the symmetric sector and the replicas are expressed by the attractive δ -Bose gas on the line.

As first shown by Mc Guire [18], the ground state energy $E_0(n)$ of H_n is given by

$$E_0(n) = -\frac{1}{24}(n^3 - n). \quad (2.12)$$

The term linear in n translates to the free energy shift $t/24$, which in fact equals the bulk free energy per unit time. For later use, we introduce the parameter

$$\alpha = (t/2)^{1/3}. \quad (2.13)$$

In the lowest order approximation, ignoring all excited states,

$$\langle 0 | e^{-tH_n} | 0 \rangle \approx e^{-tE_0(n)}. \quad (2.14)$$

The cubic term of $E_0(n)$ leads to the decay of the left tail of $F(0, t)$ as

$$\text{Prob} \left(F(0, t) - \frac{1}{24}t \leq u \right) \approx \exp \left[-\frac{4}{3} (\alpha|u|)^{3/2} \right] \quad (2.15)$$

for $u \rightarrow -\infty$ and $t^{-1/3}|u| = \mathcal{O}(1)$ [19, 20], see also [9]. This result confirms that the fluctuations of the free energy are of order $t^{1/3}$. In fact, the tail behaviour (2.15) agrees with the exact tail, see [21].

2.3. One-point generating function of the free energy

To go beyond (2.14), one needs the excited states of the attractive δ -Bose gas. They can be computed from the Bethe ansatz as has been recently worked out in great detail by Dotsenko and Klumov [22, 14]. In brackets, we remark that on a ring, the complex momenta are solutions of the nonlinear Bethe equations on which little information is available. Already to determine the ground state energy requires ingenious computations [23]. A corresponding situation has been found for the asymmetric simple exclusion process. On a ring, while the ground state and the large deviations for the current have been extensively investigated [24, 25, 26, 27], the Bethe equations for excited states have

been analyzed only partially [28, 29, 30, 31]. In contrast, for the infinite lattice there is a reasonably concise formula for the transition probability with any given number of particles [32].

Using the complete eigenfunction expansion for H_n , a particular generating function can be expressed as a Fredholm determinant [12, 13, 14]. More precisely, we define

$$G(s; x, t) = \langle \exp(-e^{-s} e^{-F(x,t)}) \rangle . \quad (2.16)$$

Then G is equal to the Fredholm determinant

$$G\left(s - \frac{1}{24}t - \frac{1}{2t}x^2; x, t\right) = \det(1 - M) . \quad (2.17)$$

The operator M does not depend on the position x and depends on time t only through the parameter α defined in (2.13). M acts on $L^2(\mathbb{R})$ and has the integral kernel

$$\langle u|M|v \rangle = \frac{e^{\alpha u - s}}{1 + e^{\alpha u - s}} \langle u|K|v \rangle . \quad (2.18)$$

Here K is the Airy operator with integral kernel

$$\langle u|K|v \rangle = \int_0^\infty dz \operatorname{Ai}(u+z) \operatorname{Ai}(z+v) , \quad (2.19)$$

called *Airy kernel*. K is related to the Airy Hamiltonian

$$H = -(\partial_u)^2 + u , \quad (2.20)$$

as K projects onto all negative eigenstates of H . In particular $K^* = K$ and $K^2 = K$, see Appendix B for details. One easily checks that $\operatorname{tr}|M| < \infty$. Hence the Fredholm determinant in (2.17) is well defined.

The mathematical status of (2.17) is somewhat tricky. One cannot simply verify (2.17) as the solution of some equation. The derivation relies on the replica method. Since $\log \langle Z(x, t)^n \rangle \approx n^3$, the moments do not uniquely determine the distribution of $Z(x, t)$. To derive (2.17), one is forced to work with divergent series and has to make a reasonable choice for the analytic extension of $\langle Z^n \rangle$, $n \in \mathbb{N}$, to the complex plane [12, 13]. However, the generating function $G(s; x, t)$ fixes the distribution of $F(x, t)$, which is known by other means [10, 8]. Thus *a posteriori* one can verify directly that (2.17) is a valid identity.

Eq. (2.17) together with (2.18) establishes that F is of order α . Rescaling s as αa and taking $\alpha \rightarrow \infty$, we note that the right side of (2.16) and the multiplicative prefactor in (2.18) both converge to a step function. Hence in the long time limit one obtains

$$\lim_{t \rightarrow \infty} \operatorname{Prob}\left(F(x, t) - \frac{1}{24}t - \frac{1}{2t}x^2 < -\alpha a\right) = \det(1 - P_a K P_a) = F_2(a) , \quad (2.21)$$

where P_a projects on $[a, \infty)$. In the long time limit, the free energy fluctuations are thus of order $t^{1/3}$. The function F_2 is the celebrated Tracy-Widom distribution function [33], which first appeared as the distribution for the maximal eigenvalue of large random Hermitian matrices in the Gaussian unitary ensemble (GUE).

By a more sophisticated argument [12] one also deduces the finite time probability density from (2.17) with the result

$$\operatorname{Prob}\left(F(x, t) - \frac{1}{24}t - \frac{1}{2t}x^2 < -\alpha s\right) = \int_{-\infty}^\infty du \exp(-e^{\alpha(s-u)}) g_t(u) , \quad (2.22)$$

where

$$g_t(u) = \det(1 - P_u(B_t - P_{\text{Ai}})P_u) - \det(1 - P_u B_t P_u) . \quad (2.23)$$

The operators P_{Ai} and B_t are defined respectively by

$$\langle z | P_{\text{Ai}} | z' \rangle = \text{Ai}(z) \text{Ai}(z') , \quad (2.24)$$

and

$$\langle z | B_t | z' \rangle = \int_{-\infty}^{\infty} dv \frac{1}{1 - e^{av}} \text{Ai}(z + v) \text{Ai}(v + z') . \quad (2.25)$$

There is another model for which the one-point generating function is available [34]: one replaces the Brownian motion by a continuous time random walk on the lattice \mathbb{Z} with forward jumps only. The white noise is correspondingly discretized in the spatial direction. The partition function is $Z_\beta(N, t)$ with the polymer starting at 0 and ending at (N, t) , $N \geq 0$. Note that this model has less scale invariance than our case because of the lattice. The long time behavior of the free energy remains to be studied.

For the directed polymer at zero temperature, results are available for a lattice discretization in case the random potential has either a one-sided exponential or geometric distribution [35].

2.4. Two-point generating function and long time limit

Our novel contribution is the extension of (2.17) to two reference points by using the replica method. In analogy to (2.16), let us define the generating function

$$G(s_1, s_2; x_1, x_2, t) = \langle \exp(-e^{-s_1} e^{-F(x_1, t)} - e^{-s_2} e^{-F(x_2, t)}) \rangle . \quad (2.26)$$

From the mapping of the directed polymer to the δ -Bose gas, the generating function G can be expanded in a sum over eigenstates of the Lieb-Liniger Hamiltonian (2.10). Using results from [14], under a natural factorization assumption the sum over the eigenstates can be written again as a Fredholm determinant. However even then it is difficult to extract any useful information from this representation. By a sequence of miraculous transformations we arrive at an alternative expression for the Fredholm determinant, which turns out to be rather similar to (2.18) in structure and from which the long time limit can be read off easily. To distinguish from the true generating function, we introduce the sharp superscript \sharp for generating functions with factorization assumption.

Let us first define the function Φ by

$$\Phi(u, v) = \frac{e^u + e^v}{1 + e^u + e^v} \quad (2.27)$$

and the operator Q through the kernel

$$\langle u | Q | v \rangle = \Phi(\alpha u - s_1, \alpha v - s_2) \langle u | e^{-(2\alpha^2)^{-1}|x_1 - x_2|H} | v \rangle . \quad (2.28)$$

Then

$$G^\sharp(s_1 - \frac{1}{24}t - \frac{1}{2t}x_1^2, s_2 - \frac{1}{24}t - \frac{1}{2t}x_2^2; x_1, x_2, t) = \det(1 - Q e^{(2\alpha^2)^{-1}|x_1 - x_2|H} K) . \quad (2.29)$$

Recall that $\langle u | e^{-(2\alpha^2)^{-1}|x_1-x_2|H} | v \rangle$ is the propagator of the Airy Hamiltonian and note that

$$\langle u | e^{(2\alpha^2)^{-1}|x_1-x_2|H} K | v \rangle = \int_0^\infty dz e^{-(2\alpha^2)^{-1}|x_1-x_2|z} \text{Ai}(u+z) \text{Ai}(z+v) . \quad (2.30)$$

In particular, $e^{(2\alpha^2)^{-1}|x_1-x_2|H} K$ is a bounded operator.

From (2.29) it is obvious that for long times F scales as $\alpha \sim t^{1/3}$ and the two-point distribution has a non-degenerate limit only if x scales as $\alpha^2 \sim t^{2/3}$. So let us substitute s_1 by αa , s_2 by αb , and introduce

$$|x_1 - x_2| = 2\alpha^2 y . \quad (2.31)$$

Then as $t \rightarrow \infty$ the left hand side of (2.29) converges to the characteristic function of the rectangle $[-\alpha a, \infty) \times [-\alpha b, \infty)$, while the operator Q converges to $P_a e^{-|y|H} + e^{-|y|H} P_b - P_a e^{-|y|H} P_b$. With y , a and b held fixed, we arrive at

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Prob} \left(F(x_1, t) - \frac{1}{24}t - \frac{1}{2t}x_1^2 > -\alpha a, F(x_2, t) - \frac{1}{24}t - \frac{1}{2t}x_2^2 > -\alpha b \right) \\ = F_2(a, b; y) , \end{aligned} \quad (2.32)$$

where the function $F_2(a, b; y)$ is given in terms of the Fredholm determinant

$$F_2(a, b; y) = \det \left(1 - (P_a + e^{-|y|H} P_b e^{|y|H} - P_a e^{-|y|H} P_b e^{|y|H}) K \right) . \quad (2.33)$$

The function $F_2(a, b; y)$ is two-point distribution of the Airy process in a form written down first in Eq. (5.8) of [21], where the two-point distribution function of the height in the polynuclear growth droplet model was studied. For the single step growth model the corresponding result was achieved in [36]. Thus apparently the factorization becomes exact in the long time limit.

As shown in [21] the function $F_2(a, b; y)$ can also be expressed as the Fredholm determinant of a 2×2 operator kernel,

$$F_2(a, b; y) = \det \left[1 - \begin{pmatrix} P_a & 0 \\ 0 & P_b \end{pmatrix} \begin{pmatrix} K & e^{-|y|H}(K-1) \\ e^{|y|H}K & K \end{pmatrix} \right] . \quad (2.34)$$

This form arises naturally when one studies the top most line in Dyson's Brownian motion. In the large N limit it converges to the Airy process, which can be viewed as the top line of an underlying extended determinantal random field. In particular, the n -point distribution is defined most directly through an operator with a $n \times n$ matrix structure. While we were searching for a corresponding matrix structure, it came as a surprise that the replica route apparently prefers the expression (2.33).

From (2.34) one can read off properties of $F_2(a, b; y)$. Obviously it is symmetric in a and b . In the limits $y \rightarrow 0$ and $y \rightarrow \infty$, the expression of $F_2(a, b; y)$ simplifies and one obtains

$$\lim_{y \rightarrow 0} F_2(a, b; y) = F_2(\min(a, b)) \quad \text{and} \quad \lim_{y \rightarrow \infty} F_2(a, b; y) = F_2(a)F_2(b) , \quad (2.35)$$

with F_2 of a single argument denoting the Tracy-Widom distribution (2.21). Using (2.33) and $[H, K] = 0$ one finds

$$\lim_{b \rightarrow -\infty} F_2(a, b; y) = 0 \quad \text{and} \quad \lim_{b \rightarrow \infty} F_2(a, b; y) = F_2(a) . \quad (2.36)$$

Since $F_2(a, b; y)$ is a distribution function, see (2.32), it is an increasing function of a, b . It tends to 0 when $a, b \rightarrow -\infty$, and to 1 when $a, b \rightarrow \infty$.

3. Replica Summation

3.1. Two-point generating function

We start from the expression (2.26) of the generating function $G(s_1, s_2; x_1, x_2, t)$ and define

$$\begin{aligned} G_1 &= G\left(s_1 - \frac{1}{24}t, s_2 - \frac{1}{24}t; x_1, x_2, t\right) \\ &= \left\langle \exp\left(-e^{t/24}e^{-s_1}Z(x_1, t) - e^{t/24}e^{-s_2}Z(x_2, t)\right) \right\rangle, \end{aligned} \quad (3.1)$$

subtracting the linear order of the free energy. Expanding the exponential, G_1 is written in terms of n -point correlations of the partition function. After decomposing as

$$X = \frac{x_1 + x_2}{2}, \quad x = \frac{x_2 - x_1}{2}, \quad (3.2)$$

we obtain

$$\begin{aligned} G_1 &= 1 + \sum_{N=1}^{\infty} \frac{(-1)^N e^{tN/24}}{N!} \sum_{\sigma_1, \dots, \sigma_N = \pm 1} \left(\prod_{i=1}^N e^{-\frac{1}{2}[(1-\sigma_i)s_1 + (1+\sigma_i)s_2]} \right) \\ &\quad \times \langle Z(X + \sigma_1 x, t) \dots Z(X + \sigma_N x, t) \rangle. \end{aligned} \quad (3.3)$$

We use the expression (2.11) for the N -point correlations in terms of the δ -Bose gas. These matrix elements can be expanded as a sum over the orthonormal basis of eigenstates, ψ_r , of the Lieb-Liniger Hamiltonian H_N with the result

$$\langle Z(x_1, t) \dots Z(x_N, t) \rangle = \sum_r e^{-tE_r} \langle x_1, \dots, x_N | \psi_r \rangle \langle \psi_r | 0 \rangle, \quad (3.4)$$

where E_r is the corresponding energy eigenvalue. Combining (3.3) and (3.4), one obtains the following expression for the generating function G_1 ,

$$\begin{aligned} G_1 &= 1 + \sum_{N=1}^{\infty} \frac{(-1)^N e^{tN/24}}{N!} \sum_r e^{-tE_r} |\psi_r(0, \dots, 0)|^2 \\ &\quad \times \sum_{\sigma_1, \dots, \sigma_N = \pm 1} \left(\prod_{i=1}^N e^{-\frac{1}{2}[(1-\sigma_i)s_1 + (1+\sigma_i)s_2]} \right) \frac{\psi_r(X + \sigma_1 x, \dots, X + \sigma_N x)}{\psi_r(0, \dots, 0)}. \end{aligned} \quad (3.5)$$

3.2. Summation over the eigenstates of the δ -Bose gas

We have to perform the summation over the eigenstates in the expression (3.5) of the generating function G_1 , for which we follow [14] with some improvements. To ease the comparison our notation will be as close as possible to the one in [14].

The eigenfunctions of the N -particle attractive δ -Bose gas are labelled by the complex wave numbers ξ_a , $a = 1, \dots, N$. To write them down, we pick positive integers n_α , $\alpha = 1, \dots, M \leq N$ such that

$$\sum_{\alpha=1}^M n_\alpha = N. \quad (3.6)$$

We also introduce the running indices $r_\alpha = 1, \dots, n_\alpha$ and set

$$n^{(\alpha)} = \sum_{\beta=1}^{\alpha} n_\beta, \quad n^{(0)} = 0, \quad n^{(M)} = N. \quad (3.7)$$

Then, for arbitrary vectors $\mathbf{q} = (q_1, \dots, q_M) \in \mathbb{R}^M$ and $\mathbf{n} = (n_1, \dots, n_M)$, one has

$$\xi_a = q_\alpha - \frac{i}{2}(n_\alpha + 1 - 2r_\alpha) \quad \text{for} \quad a = n^{(\alpha-1)} + r_\alpha, \quad (3.8)$$

with $\alpha = 1, \dots, M$. According to [14], Eq. (B.21), the eigenfunction with labels \mathbf{q}, \mathbf{n} is given by

$$\begin{aligned} & \psi_{\mathbf{q}, \mathbf{n}}^{(M)}(x_1, \dots, x_N) \\ &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_{p \in \mathcal{P}} \text{sgn}(p) \prod_{1 \leq a < b \leq N} (\xi_{p(a)} - \xi_{p(b)} + i \text{sgn}(x_a - x_b)) \exp \left[i \sum_{c=1}^N \xi_{p(c)} x_c \right]. \end{aligned} \quad (3.9)$$

The normalization constant $C_{\mathbf{q}, \mathbf{n}}^{(M)}$ is computed in [14]. The sum is over the set \mathcal{P} of all N -long permutations and $\text{sgn}(p)$ is the signature of the permutation p . $\psi_{\mathbf{q}, \mathbf{n}}^{(M)}$ is continuous and symmetric in x_1, \dots, x_N . Each eigenstate of the δ -Bose gas is made up of M clusters, where the α 's cluster consists of n_α bound particles and has center of mass momentum q_α .

In the following transformation we invoke a combinatorial identity (we owe the proof to P. Di Francesco).

Lemma 1. *Let $f(a, b)$ be arbitrary complex coefficients and let*

$$D(\xi_1, \dots, \xi_N) = \sum_{p \in \mathcal{P}} \text{sgn}(p) \prod_{1 \leq a < b \leq N} (\xi_{p(a)} - \xi_{p(b)} + f(a, b)). \quad (3.10)$$

Then D equals the Vandermonde determinant,

$$D(\xi_1, \dots, \xi_N) = N! \prod_{1 \leq a < b \leq N} (\xi_a - \xi_b). \quad (3.11)$$

Proof. D is a polynomial of total degree $N(N-1)/2$ with leading coefficient $N! \prod_{1 \leq a < b \leq N} (\xi_a - \xi_b)$. Hence, to prove (3.11) one only has to establish that D is antisymmetric, since any antisymmetric polynomial is divisible by the Vandermonde determinant.

It suffices to study the interchange of a specific pair, say ξ_1 and ξ_2 . Let p be some permutation such that $p(c) = 1$, $p(d) = 2$ and $c < d$. Then the product in (3.10) decomposes into a factor $C(p)$ independent of ξ_1, ξ_2 and a second factor as

$$\begin{aligned} & \text{sgn}(p) C(p) (\xi_1 - \xi_2 + f(c, d)) \prod_{a < c} (\xi_{p(a)} - \xi_1 + f(a, c)) \prod_{\substack{a < d \\ (a \neq c)}} (\xi_{p(a)} - \xi_2 + f(a, d)) \\ & \times \prod_{\substack{b > c \\ (b \neq d)}} (\xi_1 - \xi_{p(b)} + f(c, b)) \prod_{b > d} (\xi_2 - \xi_{p(b)} + f(d, b)). \end{aligned} \quad (3.12)$$

Let \tilde{p} be the permutation with 1 and 2 interchanged. Then $C(\tilde{p}) = C(p)$, $\text{sgn}(\tilde{p}) = -\text{sgn}(p)$, $\tilde{p}(c) = 2$, $\tilde{p}(d) = 1$ and the decomposition as in (3.12) reads

$$\begin{aligned} & -\text{sgn}(p)C(p)(\xi_2 - \xi_1 + f(c, d)) \prod_{a < c} (\xi_{p(a)} - \xi_2 + f(a, c)) \prod_{\substack{a < d \\ (a \neq c)}} (\xi_{p(a)} - \xi_1 + f(a, d)) \\ & \times \prod_{\substack{b > c \\ (b \neq d)}} (\xi_2 - \xi_{p(b)} + f(c, b)) \prod_{b > d} (\xi_1 - \xi_{p(b)} + f(d, b)) . \end{aligned} \quad (3.13)$$

It is now obvious that the sum of the two terms is antisymmetric in ξ_1 and ξ_2 , and hence $D(\xi_1, \xi_2, \dots, \xi_N) = -D(\xi_2, \xi_1, \dots, \xi_N)$. \square

We first compute $\psi_{\mathbf{q}, \mathbf{n}}^{(M)}(0)$ by taking the limit $\varepsilon \rightarrow 0$ of $x_j = \varepsilon j$. Then $\text{sgn}(x_a - x_b) = -1$ for all $a < b$ and by Lemma 1

$$\psi_{\mathbf{q}, \mathbf{n}}^{(M)}(0) = C_{\mathbf{q}, \mathbf{n}}^{(M)} N! \prod_{1 \leq a < b \leq N} (\xi_a - \xi_b) . \quad (3.14)$$

Secondly we have to evaluate (3.9) at $x_j = X + x\sigma_j$ and to perform the sum over all “spin” configurations $\boldsymbol{\sigma} = \{\sigma_1, \dots, \sigma_N\}$, for which purpose we use that (3.9) can be written more compactly by using the special structure of the complex wave numbers, see [14], Sect. B.2. We introduce the cluster counting function $\alpha : [1, \dots, N] \rightarrow [1, \dots, M]$ by

$$\alpha(a) = \beta \quad \text{for} \quad n^{(\beta-1)} < a \leq n^{(\beta)} , \quad \beta = 1, \dots, M , \quad (3.15)$$

and the β -th cluster by

$$\Omega_\beta(p) = \{a \mid \alpha(p(a)) = \beta\} . \quad (3.16)$$

Then, working out the derivatives in [14], Eq. (B.28),

$$\begin{aligned} \psi_{\mathbf{q}, \mathbf{n}}^{(M)}(x_1, \dots, x_N) &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum'_{p \in \mathcal{P}} \text{sgn}(p) \prod_{\substack{1 \leq a, b \leq N \\ \alpha(p(a)) \neq \alpha(p(b))}} (q_{\alpha(p(a))} - q_{\alpha(p(b))} + i\eta(x_a, x_b)) \\ &\times \exp \left[i \sum_{\alpha=1}^M q_\alpha \sum_{c \in \Omega_\alpha(p)} x_c - \frac{1}{4} \sum_{\alpha=1}^M \sum_{c, c' \in \Omega_\alpha(p)} |x_c - x_{c'}| \right] . \end{aligned} \quad (3.17)$$

Here the sum over permutations is understood modulo permutations inside each cluster, as indicated by $'$, and η is defined by

$$\eta(x_a, x_b) = \text{sgn}(x_a - x_b) + \frac{1}{2} \left(\sum_{\substack{c \in \Omega_\alpha(p) \\ c \neq a \in \Omega_\alpha(p)}} \text{sgn}(x_a - x_c) - \sum_{\substack{c \in \Omega_\alpha(p) \\ c \neq b \in \Omega_\alpha(p)}} \text{sgn}(x_b - x_c) \right) . \quad (3.18)$$

We spread the particle configuration as

$$x_a = X + x\sigma_a + \varepsilon a . \quad (3.19)$$

Then

$$\begin{aligned} & \sum_{\boldsymbol{\sigma}} \left(\prod_{i=1}^N e^{-\frac{1}{2}[(1-\sigma_i)s_1 + (1+\sigma_i)s_2]} \right) \psi_{\mathbf{q}, \mathbf{n}}^{(M)}(X + \sigma_1 x + \varepsilon, \dots, X + \sigma_N x + N\varepsilon) \\ &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_{\boldsymbol{\sigma}} \sum'_{p \in \mathcal{P}} \text{sgn}(p) \prod_{\substack{1 \leq a, b \leq N \\ \alpha(p(a)) \neq \alpha(p(b))}} (q_{\alpha(p(a))} - q_{\alpha(p(b))} + i\eta(x_a, x_b)) e^{\phi(\boldsymbol{\sigma}, p)} . \end{aligned} \quad (3.20)$$

The phase $\phi(\boldsymbol{\sigma}, p)$ is given by

$$\begin{aligned} \phi(\boldsymbol{\sigma}, p) = \sum_{\alpha=1}^M \Big(& -\frac{1}{2}(s_1 + s_2)n_\alpha + \frac{1}{2}(s_1 - s_2)m_\alpha(\boldsymbol{\sigma}, p) \\ & + i q_\alpha(X n_\alpha + x m_\alpha(\boldsymbol{\sigma}, p)) - \frac{1}{2}|x|(n_\alpha^2 - m_\alpha(\boldsymbol{\sigma}, p)^2) \Big), \end{aligned} \quad (3.21)$$

where we introduced

$$m_\alpha(\boldsymbol{\sigma}, p) = \sum_{c \in \Omega_\alpha(p)} \sigma_c. \quad (3.22)$$

Inserting in (3.20) one arrives at

$$\begin{aligned} & \sum_{\boldsymbol{\sigma}} \left(\prod_{i=1}^N e^{-\frac{1}{2}[(1-\sigma_i)s_1 + (1+\sigma_i)s_2]} \right) \psi_{\mathbf{q}, \mathbf{n}}^{(M)}(X + \sigma_1 x + \varepsilon, \dots, X + \sigma_N x + N\varepsilon) \\ &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_{\boldsymbol{\sigma}} \sum'_{p \in \mathcal{P}} \text{sgn}(p) \prod_{\substack{1 \leq a, b \leq N \\ \alpha(p(a)) \neq \alpha(p(b))}} (q_{\alpha(p(a))} - q_{\alpha(p(b))} + i\eta(\sigma_a + \varepsilon a, \sigma_b + \varepsilon b)) \\ & \times \prod_{\alpha=1}^M \prod_{c \in \Omega_\alpha(p)} \exp \left[-\frac{1}{2}(s_1 + s_2)n_\alpha + \frac{1}{2}(s_1 - s_2)m_\alpha(\boldsymbol{\sigma}, p) \right. \\ & \quad \left. + i q_\alpha(X n_\alpha + x m_\alpha(\boldsymbol{\sigma}, p)) - \frac{1}{2}|x|(n_\alpha^2 - m_\alpha(\boldsymbol{\sigma}, p)^2) \right]. \end{aligned} \quad (3.23)$$

Let us shorthand the right hand side of Eq. (3.23) as

$$\sum_{\boldsymbol{\sigma}} \sum_{p \in \mathcal{P}} f_1(\boldsymbol{\sigma}, p) f_2(\boldsymbol{\sigma}, p). \quad (3.24)$$

By Lemma 1

$$\sum_{p \in \mathcal{P}} f_1(\boldsymbol{\sigma}, p) = \psi_{\mathbf{q}, \mathbf{n}}^{(M)}(0) \quad (3.25)$$

not depending on $\boldsymbol{\sigma}$ and, since $f_2(\boldsymbol{\sigma}, p)$ depends on $\boldsymbol{\sigma}$ only through the $m_\alpha(\boldsymbol{\sigma}, p)$'s, correspondingly

$$\sum_{\boldsymbol{\sigma}} f_2(\boldsymbol{\sigma}, p) = \tilde{c} \quad (3.26)$$

with \tilde{c} not depending on p . Unfortunately we could not discover any further simplification. To proceed anyhow a natural step is to factorize (3.23) either with respect to p or with respect to $\boldsymbol{\sigma}$, both leading in approximation to

$$\begin{aligned} & \sum_{\boldsymbol{\sigma}} \left(\prod_{i=1}^N e^{-\frac{1}{2}[(1-\sigma_i)s_1 + (1+\sigma_i)s_2]} \right) (\psi_{\mathbf{q}, \mathbf{n}}^{(M)}(X + \sigma_1 x + \varepsilon, \dots, X + \sigma_N x + N\varepsilon) / \psi_{\mathbf{q}, \mathbf{n}}^{(M)}(0)) \\ & \simeq \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^M \exp \left[-\frac{1}{2}(s_1 + s_2)n_\alpha + \frac{1}{2}(s_1 - s_2)m_\alpha(\boldsymbol{\sigma}) \right. \\ & \quad \left. + i q_\alpha(X n_\alpha + x m_\alpha(\boldsymbol{\sigma})) - \frac{1}{2}|x|(n_\alpha^2 - m_\alpha(\boldsymbol{\sigma})^2) \right], \end{aligned} \quad (3.27)$$

where

$$m_\alpha(\boldsymbol{\sigma}) = \sum_{r_\alpha=1}^{n_\alpha} \sigma_{n^{(\alpha-1)}+r_\alpha}. \quad (3.28)$$

For small cluster sizes we checked that (3.27) is indeed not a strict equality.

4. Two-point generating function

4.1. Linearization and “spin” summation

We linearize the terms quadratic in n_α and in $m_\alpha(\boldsymbol{\sigma})$ in the exponential of (3.27), so to be able to perform the summation over the σ_i ’s and the n_α ’s. For this purpose we use the identity

$$e^{au+bv+cv} = e^{c\partial_a\partial_b} e^{au+bv}, \quad (4.1)$$

which can be checked by expanding both sides of the equation as a formal power series in c , and obtain

$$\begin{aligned} & \sum_{\boldsymbol{\sigma}} \left(\prod_{i=1}^N e^{-\frac{1}{2}[(1-\sigma_i)s_1+(1+\sigma_i)s_2]} \right) \psi_{\mathbf{q},\mathbf{n}}^{(M)}(X + \sigma_1 x, \dots, X + \sigma_N x) / \psi_{\mathbf{q},\mathbf{n}}^{(M)}(0) \\ & \simeq \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^M e^{-x\partial_1\partial_2} \exp \left[-\frac{s_1}{2}(n_\alpha - m_\alpha(\boldsymbol{\sigma})) - \frac{s_2}{2}(n_\alpha + m_\alpha(\boldsymbol{\sigma})) + iq_\alpha(Xn_\alpha + xm_\alpha(\boldsymbol{\sigma})) \right]. \end{aligned} \quad (4.2)$$

Here we introduced the convention

$$\partial_1 = \partial_{s_1}, \quad \partial_2 = \partial_{s_2}, \quad (4.3)$$

which will be used onwards. We note that the exponential inside the product over α depends only on the σ_a with $n^{(\alpha-1)} < a \leq n^{(\alpha)}$. Thus, the summation over the σ_a can be performed independently for each α . Recalling (3.2), we find

$$\begin{aligned} & \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^M e^{-x\partial_1\partial_2} \exp \left[-\frac{s_1}{2}(n_\alpha - m_\alpha(\boldsymbol{\sigma})) - \frac{s_2}{2}(n_\alpha + m_\alpha(\boldsymbol{\sigma})) + iq_\alpha(Xn_\alpha + xm_\alpha(\boldsymbol{\sigma})) \right] \\ & = \prod_{\alpha=1}^M e^{-x\partial_1\partial_2} \left(e^{ix_1q_\alpha-s_1} + e^{ix_2q_\alpha-s_2} \right)^{n_\alpha}. \end{aligned} \quad (4.4)$$

4.2. Fredholm determinant

We now return to the generating function G_1 of (3.5), denoting by G_1^\sharp its approximation under the factorization assumption (3.27). The eigenfunctions $\psi_{\mathbf{q},\mathbf{n}}^{(M)}$ are normalized in such a way that they form an orthonormal basis in the symmetric subspace of $L^2(\mathbb{R}^N)$. With this normalization, and using (3.14),

$$|\psi_{\mathbf{q},\mathbf{n}}^{(M)}(0, \dots, 0)|^2 = N! \det \left(\frac{1}{\frac{1}{2}(n_j + n_k) + i(q_j - q_k)} \right)_{j,k=1,\dots,M}, \quad (4.5)$$

see Eqs. (B.58) and (34) of [14]. From Eq. (B.29) of [14], the energy $E_{\mathbf{q},\mathbf{n}}^{(M)}$ of the eigenstate $\psi_{\mathbf{q},\mathbf{n}}^{(M)}$ is

$$E_{\mathbf{q},\mathbf{n}}^{(M)} = \frac{1}{2} \sum_{j=1}^M n_j q_j^2 - \frac{1}{24} \sum_{j=1}^M (n_j^3 - n_j). \quad (4.6)$$

Finally, the properly normalized sum over the eigenstates is given by

$$\sum_r \equiv \sum_{M=1}^{\infty} \frac{1}{M!} \prod_{j=1}^M \left(\int_{-\infty}^{\infty} \frac{dq_j}{2\pi} \sum_{n_j=1}^{\infty} \right) \mathbb{1}_{\{n=\sum_{j=1}^M n_j\}} , \quad (4.7)$$

see Eqs. (B.53) and (B.60) of [14]. Combining all, we obtain for G_1^\sharp the following expression

$$\begin{aligned} G_1^\sharp = 1 + \sum_{M=1}^{\infty} \frac{1}{M!} \prod_{j=1}^M \left(\int_{-\infty}^{\infty} \frac{dq_j}{2\pi} \sum_{n_j=1}^{\infty} \right) \det \left(\frac{1}{\frac{1}{2}(n_j + n_k) + i(q_j - q_k)} \right)_{j,k=1,\dots,M} \\ \times \prod_{j=1}^M e^{tn_j^3/24} e^{-x\partial_1\partial_2} \left[-e^{-tq_j^2/2} \left(e^{ix_1q_j-s_1} + e^{ix_2q_j-s_2} \right) \right]^{n_j} . \end{aligned} \quad (4.8)$$

We observe that (4.8) can be rewritten as a Fredholm determinant (see Appendix A for a few basic facts about Fredholm determinants). Indeed, if one introduces the kernel R as

$$R(q, n; q', n') = \frac{1}{2\pi} \frac{e^{tn^3/24} e^{-x\partial_1\partial_2} \left[-e^{-\frac{t}{2}q^2} \left(e^{ix_1q-s_1} + e^{ix_2q-s_2} \right) \right]^n}{\frac{1}{2}(n + n') + i(q - q')} , \quad (4.9)$$

then the generating function G_1^\sharp is given by

$$G_1^\sharp = \det(1 + R) . \quad (4.10)$$

More explicitly, it holds

$$G_1^\sharp = 1 + \sum_{M=1}^{\infty} \frac{1}{M!} \int_{-\infty}^{\infty} dq_1 \dots dq_M \sum_{n_1, \dots, n_M=1}^{\infty} \det(R(q_j, n_j; q_k, n_k))_{j,k=1,\dots,M} . \quad (4.11)$$

We perform the summation over the n_j 's and the integration over the q_j 's. For this purpose the integrated version for the denominator in (4.9) is used,

$$\frac{1}{\frac{1}{2}(n + n') + i(q - q')} = \int_0^\infty dz e^{-z[\frac{1}{2}(n+n') + i(q-q')]} . \quad (4.12)$$

The operator R can then be written as a product of two operators, $R = R_1 R_2$,

$$R(q, n; q', n') = \int_{-\infty}^{\infty} dz R_1(q, n; z) R_2(z; q', n') , \quad (4.13)$$

with

$$R_1(q, n; z) = \mathbb{1}_{\{z>0\}} e^{-iqz} e^{tn^3/24} e^{-x\partial_1\partial_2} \left[-e^{-\frac{1}{2}z} e^{-tq^2/2} \left(e^{ix_1q-s_1} + e^{ix_2q-s_2} \right) \right]^n , \quad (4.14)$$

and

$$R_2(z; q', n') = \mathbb{1}_{\{z>0\}} \frac{1}{2\pi} e^{-\frac{1}{2}n'z} e^{iq'z} . \quad (4.15)$$

Using $\det(1 + R_1 R_2) = \det(1 + R_2 R_1)$, the generating function G_1^\sharp becomes equal to a Fredholm determinant of the new operator N ,

$$G_1^\sharp = \det(1 + N) , \quad (4.16)$$

where $N = R_2 R_1$ with kernel

$$N(z, z') = \int_{-\infty}^{\infty} dq \sum_{n=1}^{\infty} R_2(z; q, n) R_1(q, n; z') . \quad (4.17)$$

The variables q_j and n_j , which were previously the variables corresponding to the definition of the Fredholm determinant, are now inside the kernel N .

In Appendix C, the summation over n and the integration over q in (4.17) is performed explicitly. Most of the steps are rather similar to the computations done in [13, 12, 14] in case of the one-point generating function. Note that on face value the sum over n is badly divergent. In terms of the parameter $\alpha = (t/2)^{1/3}$ (equal to $2^{2/3}\lambda$ in the notation of [14]) and of the function Φ defined in equation (2.27), the kernel of N equals

$$N(z, z') = -\alpha^{-1} \tilde{N}(\alpha^{-1}z, \alpha^{-1}z') , \quad (4.18)$$

with

$$\tilde{N}(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du e^{-x\partial_1\partial_2} e^{-(2\alpha)^{-1}(x_1\partial_1+x_2\partial_2)(\partial_z-\partial_{z'})} \Phi(\alpha u - s_1, \alpha u - s_2) \text{Ai}(z+u) \text{Ai}(u+z') . \quad (4.19)$$

The generating function G_1^\sharp is now given by

$$G_1^\sharp = \det(1 - \tilde{N}) . \quad (4.20)$$

We will simplify the kernel \tilde{N} and express it in terms of the Airy Hamiltonian H and of the Airy operator K .

4.3. Subtraction of the parabolic profile

The following transformations are guided to have the shift by $x^2/2t$ manifestly visible in G_1^\sharp . Using the definition (3.2) of X and x , the expression (4.19) of the kernel \tilde{N} rewrites as

$$\tilde{N}(z, z') = \mathbb{1}_{\{z, z' > 0\}} e^{-x\partial_1\partial_2} e^{\frac{x}{2\alpha}(\partial_1-\partial_2)(\partial_z-\partial_{z'})} \int_{-\infty}^{\infty} du e^{-\frac{X}{2\alpha}(\partial_1+\partial_2)(\partial_z-\partial_{z'})} \Phi(\alpha u - s_1, \alpha u - s_2) \text{Ai}(z+u) \text{Ai}(u+z') . \quad (4.21)$$

We note that $\partial_1 + \partial_2$ in (4.21) can be replaced by $-\alpha^{-1}\partial_u$, where the derivative ∂_u acts only on $\Phi(\alpha u - s_1, \alpha u - s_2)$ and not on the product of Airy functions. One can then make ∂_u to act only on the product of Airy functions by integrating by parts. Thereby

$$\tilde{N}(z, z') = \mathbb{1}_{\{z, z' > 0\}} e^{-x\partial_1\partial_2} e^{\frac{x}{2\alpha}(\partial_1-\partial_2)(\partial_z-\partial_{z'})} \int_{-\infty}^{\infty} du \Phi(\alpha u - s_1, \alpha u - s_2) e^{-(2\alpha^2)^{-1}X\partial_u(\partial_z-\partial_{z'})} \text{Ai}(z+u) \text{Ai}(u+z') . \quad (4.22)$$

Since the Airy function is a solution of the differential equation $\text{Ai}''(u) = u\text{Ai}(u)$, we have

$$\partial_u(\partial_z - \partial_{z'}) \text{Ai}(z+u) \text{Ai}(u+z') = (z - z') \text{Ai}(z+u) \text{Ai}(u+z') \quad (4.23)$$

and hence

$$e^{a[\partial_u(\partial_z - \partial_{z'}) - (z - z')]} \text{Ai}(z + u) \text{Ai}(u + z') = \text{Ai}(z + u) \text{Ai}(u + z') . \quad (4.24)$$

The commutator of $\partial_u(\partial_z - \partial_{z'})$ and $(z - z')$ is equal to $2\partial_u$ and it commutes with both $\partial_u(\partial_z - \partial_{z'})$ and $(z - z')$. Thus the Baker-Campbell-Hausdorff formula terminates and

$$e^{-a\partial_u(\partial_z - \partial_{z'})} \text{Ai}(z + u) \text{Ai}(u + z') = e^{-a(z - z')} e^{a^2\partial_u} \text{Ai}(z + u) \text{Ai}(u + z') . \quad (4.25)$$

We use this property in the expression of the kernel \tilde{N} , and integrate by parts to make ∂_u act again on $\Phi(\alpha u - s_1, \alpha u - s_2)$ with the result

$$\begin{aligned} \tilde{N}(z, z') &= \mathbb{1}_{\{z, z' > 0\}} e^{-x\partial_1\partial_2} e^{\frac{x}{2\alpha}(\partial_1 - \partial_2)(\partial_z - \partial_{z'})} \int_{-\infty}^{\infty} du e^{-(2\alpha^2)^{-1}X(z - z')} \text{Ai}(z + u) \\ &\quad \times \text{Ai}(u + z') e^{-(4\alpha^4)^{-1}X^2\partial_u} \Phi(\alpha u - s_1, \alpha u - s_2) . \end{aligned} \quad (4.26)$$

Using the commutation relation

$$e^{a(\partial_z - \partial_{z'})} e^{-b(z - z')} = e^{-2ab} e^{-b(z - z')} e^{a(\partial_z - \partial_{z'})} , \quad (4.27)$$

one obtains

$$\begin{aligned} \tilde{N}(z, z') &= \mathbb{1}_{\{z, z' > 0\}} e^{-(2\alpha^2)^{-1}X(z - z')} e^{-x\partial_1\partial_2} e^{\frac{x}{2\alpha}(\partial_1 - \partial_2)(\partial_z - \partial_{z'})} \int_{-\infty}^{\infty} du \text{Ai}(z + u) \\ &\quad \times \text{Ai}(u + z') e^{-(2\alpha^3)^{-3}Xx(\partial_1 - \partial_2)} e^{-(4\alpha^4)^{-1}X^2\partial_u} \Phi(\alpha u - s_1, \alpha u - s_2) . \end{aligned} \quad (4.28)$$

Since $e^{a\partial}$ acts as a shift operator, we arrive at

$$\begin{aligned} \tilde{N}(z, z') &= \mathbb{1}_{\{z, z' > 0\}} e^{-(2\alpha^2)^{-1}X(z - z')} e^{-x\partial_1\partial_2} e^{\frac{x}{2\alpha}(\partial_1 - \partial_2)(\partial_z - \partial_{z'})} e^{-(4\alpha^3)^{-1}x_2^2(\partial_1 + \partial_2)} \\ &\quad \times \int_{-\infty}^{\infty} du \text{Ai}(z + u) \text{Ai}(u + z') \Phi\left(\alpha u - s_1 - \frac{1}{2t}x_1^2, \alpha u - s_2 - \frac{1}{2t}x_2^2\right) . \end{aligned} \quad (4.29)$$

The factor $\exp(-(2\alpha^2)^{-1}X(z - z'))$ can be eliminated by a similarity transformation of the kernel, it does not contribute to the Fredholm determinant. We define the shifted generating function

$$G_2^\sharp(u, v; x, t) = G_1^\sharp(u - \frac{1}{2t}x_1^2, v - \frac{1}{2t}x_2^2; x_1, x_2, t) . \quad (4.30)$$

and find that G_2^\sharp can be written as

$$G_2^\sharp = \det(1 - L) , \quad (4.31)$$

where the kernel L is given by

$$\begin{aligned} L(z, z') &= \mathbb{1}_{\{z, z' > 0\}} e^{-x\partial_1\partial_2} e^{\frac{x}{2\alpha}(\partial_1 - \partial_2)(\partial_z - \partial_{z'})} e^{-(4\alpha^3)^{-1}x^2(\partial_1 + \partial_2)} \\ &\quad \times \int_{-\infty}^{\infty} du \text{Ai}(z + u) \text{Ai}(u + z') \Phi(\alpha u - s_1, \alpha u - s_2) . \end{aligned} \quad (4.32)$$

In this form, the kernel depends only on x (and t), as to be expected from the discussion in the introduction.

4.4. Rewriting of the kernel L in terms of the Airy Hamiltonian

The next step is to eliminate the operator $\partial_1 \partial_2$ from the expression (4.32) for the kernel L , for which purpose one writes

$$L(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du e^{\frac{x}{2}(\partial_1^2 + \partial_2^2)} e^{-\frac{x}{2}(\partial_1 + \partial_2)^2} e^{-(4\alpha^3)^{-1}x^2(\partial_1 + \partial_2)} e^{\frac{x}{2\alpha}(\partial_1 - \partial_2)(\partial_z - \partial_{z'})} \\ \times \Phi(\alpha u - s_1, \alpha u - s_2) \text{Ai}(z + u) \text{Ai}(u + z'). \quad (4.33)$$

In this expression, we can replace $(\partial_1 + \partial_2)^2$ by $\alpha^{-2} \partial_u^2$, where ∂_u acts only on $\Phi(\alpha u - s_1, \alpha u - s_2)$ and not on the product of Airy functions. Then, integrating by parts, ∂_u acts on the product of Airy functions instead. Thereby

$$L(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du e^{\frac{x}{2}(\partial_1^2 + \partial_2^2)} e^{-(4\alpha^3)^{-1}x^2(\partial_1 + \partial_2)} e^{\frac{x}{2\alpha}(\partial_1 - \partial_2)(\partial_z - \partial_{z'})} \\ \times \Phi(\alpha u - s_1, \alpha u - s_2) e^{-(2\alpha^2)^{-1}x\partial_u^2} \text{Ai}(z + u) \text{Ai}(u + z'). \quad (4.34)$$

But ∂_u acting on $\text{Ai}(z + u) \text{Ai}(u + z')$ is the same as $\partial_z + \partial_{z'}$. Thus, one can replace $\exp(-(2\alpha^2)^{-1}x\partial_u^2)$ by $\exp(-(2\alpha^2)^{-1}x\partial_u(\partial_z + \partial_{z'}))$. After integrating by parts again, we replace $\exp((2\alpha^2)^{-1}x\partial_u(\partial_z + \partial_{z'}))$ by $\exp(-\frac{x}{2\alpha}(\partial_1 + \partial_2)(\partial_z + \partial_{z'}))$ and obtain

$$L(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du e^{\frac{x}{2}(\partial_1^2 + \partial_2^2)} e^{-(4\alpha^3)^{-1}x^2(\partial_1 + \partial_2)} e^{-\frac{x}{\alpha}(\partial_1 \partial_{z'} + \partial_2 \partial_z)} \\ \times \Phi(\alpha u - s_1, \alpha u - s_2) \text{Ai}(z + u) \text{Ai}(u + z'). \quad (4.35)$$

We now introduce a dummy integration to achieve that $\alpha u - s_1, \alpha u - s_2, \text{Ai}(z + u)$ and $\text{Ai}(u + z')$ do not all depend on the same variable u . It holds

$$L(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du dv \delta(u - v) e^{\frac{x}{2}(\partial_1^2 - \frac{2}{\alpha} \partial_1 \partial_{z'})} e^{\frac{x}{2}(\partial_2^2 - \frac{2}{\alpha} \partial_2 \partial_z)} e^{-(4\alpha^3)^{-1}x^2(\partial_1 + \partial_2)} \\ \times \Phi(\alpha u - s_1, \alpha v - s_2) \text{Ai}(z + v) \text{Ai}(u + z'). \quad (4.36)$$

Note that

$$[(\partial_1^2 - 2\alpha^{-1} \partial_1 \partial_{z'}) + \alpha^{-2}(z' + H_u)] \Phi(\alpha u - s_1, \alpha v - s_2) \text{Ai}(z + v) \text{Ai}(u + z') = 0, \quad (4.37)$$

where $H_u = -\partial_u^2 + u$ denotes the Airy Hamiltonian acting on the variable u . The commutation relation

$$[(\partial_1^2 - 2\alpha^{-1} \partial_1 \partial_{z'}), \alpha^{-2}(z' + H_u)] = -2\alpha^{-3} \partial_1 \quad (4.38)$$

together with the Baker-Campbell-Hausdorff formula implies

$$e^{\frac{x}{2}(\partial_1^2 - \frac{2}{\alpha} \partial_1 \partial_{z'})} \Phi(\alpha u - s_1, \alpha v - s_2) \text{Ai}(z + v) \text{Ai}(u + z') \\ = e^{-(2\alpha^2)^{-1}x(z' + H_u)} e^{(4\alpha^3)^{-1}x^2 \partial_1} \Phi(\alpha u - s_1, \alpha v - s_2) \text{Ai}(z + v) \text{Ai}(u + z'). \quad (4.39)$$

A corresponding relation is obtained by interchanging the roles of z, s_1, u and of z', s_2, v . Using both the kernel L becomes

$$L(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du dv \delta(u - v) e^{-(2\alpha^2)^{-1}x(z + z')} e^{-(2\alpha^2)^{-1}x(H_u + H_v)} \\ \times \Phi(\alpha u - s_1, \alpha v - s_2) \text{Ai}(z + v) \text{Ai}(u + z'). \quad (4.40)$$

A final integration by parts over u and v yields

$$L(z, z') = \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du dv \langle u | e^{-\alpha^{-2} x H} | v \rangle \\ \times \Phi(\alpha u - s_1, \alpha v - s_2) e^{-(2\alpha^2)^{-1} x(z+z')} \text{Ai}(z+v) \text{Ai}(u+z') . \quad (4.41)$$

This last expression is an operator product of the form ABA. Hence using the cyclicity of the determinant one arrives at

$$G_2^\sharp = \det(1 - L) = \det(1 - \tilde{L}) , \quad (4.42)$$

where

$$\langle u | \tilde{L} | v \rangle = \langle u | e^{-\alpha^{-2} x H} | v \rangle \Phi(\alpha u - s_1, \alpha v - s_2) \langle u | e^{\alpha^{-2} x H} K | v \rangle . \quad (4.43)$$

We conclude that $\tilde{L} = Q e^{\alpha^{-2} |x| H} K$, as to be shown.

5. Finite time probability density function

The two-point free energy fluctuations can be written as

$$F(x_j, t) = \frac{1}{24} t + \frac{1}{2t} x_j^2 + \xi_j(t), \quad j = 1, 2 , \quad (5.1)$$

with random amplitudes $\xi_j(t)$. In the factorization approximation, we know already that $\xi_j(t) = \mathcal{O}(t^{1/3})$. The joint distribution depends only on $|x_1 - x_2|$, and nondegenerate correlations occur for a separation of order $t^{2/3}$. Following the procedure in [12], we would like to extract the underlying pdf from the generating function G^\sharp . Let us denote by $\rho_t^\sharp(w_1, w_2)$ the approximate joint pdf of $\xi_1(t)$ and $\xi_2(t)$, and write it as the convolution with two independent Gumbel densities F'_{Gu} :

$$\rho_t^\sharp(w_1, w_2) = \int_{-\infty}^{\infty} dv_1 dv_2 F'_{\text{Gu}}(w_1 - v_1) F'_{\text{Gu}}(w_2 - v_2) g_t(v_1, v_2) , \quad (5.2)$$

where

$$F_{\text{Gu}}(w) = \exp(-e^{-w}) , \quad (5.3)$$

compare with the one-point distribution (2.22). Eq. (5.2) defines the yet to be determined function g_t , which is normalized to 1 by construction. From numerical solutions in the one-point case, we know that g_t of (2.23) is in general not everywhere positive. This implies that g_t of (5.2) will not be a pdf, in general.

The generating function G_2^\sharp reads

$$G_2^\sharp = \int_{-\infty}^{\infty} dw_1 dw_2 \rho_t(w_1, w_2) \exp(-e^{-s_1 - w_1} - e^{-s_2 - w_2}) . \quad (5.4)$$

Inserting (5.2) and performing the integration over w_1 and w_2 yields

$$G_2^\sharp = \int_{-\infty}^{\infty} dv_1 dv_2 g_t(v_1, v_2) \frac{e^{v_1}}{e^{v_1} + e^{-s_1}} \frac{e^{v_2}}{e^{v_2} + e^{-s_2}} . \quad (5.5)$$

We analytically continue on both sides from e^{-s_j} to $-e^{a_j} - i\sigma_j\varepsilon$, $\varepsilon > 0$, $j = 1, 2$. From the identity

$$\lim_{\varepsilon \rightarrow 0} \sum_{\sigma=\pm 1} \frac{\sigma e^v}{e^v - e^a - i\sigma\varepsilon} = 2i\pi\delta(v - a) , \quad (5.6)$$

the left side of (5.5) multiplied by $\sigma_1\sigma_2$ and summed over $\sigma_1, \sigma_2 = \pm 1$ yields in the limit $\varepsilon \rightarrow 0$

$$-4\pi^2 g_t(a_1, a_2) . \quad (5.7)$$

On the right side of (5.5), using $G_2^\# = \det(1 - L)$ with L given by (4.41), we obtain the sum of four Fredholm determinants with operators L_{σ_1, σ_2} , $\sigma_j = \pm 1$. To compute these kernels, we have to take the limit $\varepsilon \rightarrow 0$ of

$$\int_{-\infty}^{\infty} du_1 du_2 h(u_1, u_2) \frac{-e^{\alpha u_1 + a_1} - e^{\alpha u_2 + a_2} - i\varepsilon(\sigma_1 e^{\alpha u_1} + \sigma_2 e^{\alpha u_2})}{1 - e^{\alpha u_1 + a_1} - e^{\alpha u_2 + a_2} - i\varepsilon(\sigma_1 e^{\alpha u_1} + \sigma_2 e^{\alpha u_2})} , \quad (5.8)$$

with h general at this stage. Using

$$\frac{1}{y - i\sigma\varepsilon} = \mathcal{P}\left(\frac{1}{y}\right) + i\sigma\pi\delta(y) , \quad (5.9)$$

one obtains

$$\begin{aligned} L_{\sigma_1, \sigma_2}(z, z') &= \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du dv \langle u | e^{-\alpha^{-2} x H} | v \rangle e^{-(2\alpha^2)^{-1} x(z+z')} \text{Ai}(z+u) \text{Ai}(v+z') \\ &\quad \times \left(\mathcal{P}\left(\frac{-e^{\alpha u_1 + a_1} - e^{\alpha u_2 + a_2}}{1 - e^{\alpha u_1 + a_1} - e^{\alpha u_2 + a_2}}\right) + i\pi \text{sgn}(\sigma_1 e^{\alpha u_1} + \sigma_2 e^{\alpha u_2}) \right. \\ &\quad \left. \times (-e^{\alpha u_1 + a_1} - e^{\alpha u_2 + a_2}) \delta(1 - e^{\alpha u_1 + a_1} - e^{\alpha u_2 + a_2}) \right) , \end{aligned} \quad (5.10)$$

and

$$g_t(a_1, a_2) = -(2\pi)^{-2} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \det(1 - L_{\sigma_1, \sigma_2}) . \quad (5.11)$$

In the one-point case, inserting the corresponding analytic continuation in (2.18) yields a one-dimensional projection. This further simplifies the expression for the pdf, compare with (2.23). For the two-point case, no further simplification seems to be available.

6. Conclusions

Recently the probability distribution of the free energy of the point-to-point continuum directed polymer has been computed exactly, using the approximation through the weakly asymmetric simple exclusion process [10, 11]. This result could be reproduced by using replicas and the complete eigenfunction expansion of the propagator of the attractive δ -Bose gas on the line [12, 13].

In our contribution we studied the joint pdf of $F(x_1, t), F(x_2, t)$. In this case the approximation through the weakly asymmetric exclusion process, while still valid, is no longer computable and we have to rely on the replica method, which yields a

particular generating function. Invoking a specific factorization, the result is expressed as a Fredholm determinant. In fact, the corresponding operator has a structure rather similar to the case of one point, compare (2.17) and (2.29).

Equipped with this information, we established the large time limit of the pdf yielding a result in agreement with lattice directed polymers at zero temperature. In 1+1 dimensions all models with a short range disorder potential are expected to flow to the zero temperature fixed point. Our result further supports this claim.

The free energy of the point-to-point continuum directed polymer is isomorphic to the solution of the KPZ equation with sharp wedge initial data. Thus we have automatically determined the joint pdf of the heights $h(x_1, t)$, $h(x_2, t)$ of the KPZ equation for large times, in particular the height-height correlation function. This function has been measured recently for droplet growth in a thin film of turbulent liquid crystal [37]. The experimental curve agrees very well with the theoretical prediction in the limit $t \rightarrow \infty$.

Acknowledgements

It is a pleasure to thank Pierre Le Doussal, Michael Prähofer, and Tomohiro Sasamoto for constructive discussions.

Appendix A. Fredholm determinants

Let us first consider the case of a finite $n \times n$ matrix A . The Taylor expansion of the determinant of $1 + zA$ is given by the von Koch formula [38] in terms of the minors of A . It holds

$$\det(1 + zA) = \sum_{m=0}^n \frac{z^m}{m!} \sum_{i_1, \dots, i_m=1}^n \det(A_{i_j, i_k})_{1 \leq j, k \leq m}. \quad (\text{A.1})$$

If the matrix A is now replaced by an integral operator A with kernel $A(u, v) = \langle u|A|v \rangle$, von Koch formula (A.1) formally rewrites as

$$\det(1 + zA) \equiv \sum_{m=0}^{\infty} \frac{z^m}{m!} \int du_1 \dots du_m \det(A(u_j, u_k))_{1 \leq j, k \leq m}. \quad (\text{A.2})$$

Of course, du could mean a more general summation procedure. In particular, it could refer to summation over some discrete index and integration over \mathbb{R} . (A.2) can be used as the *definition* of the Fredholm determinant $\det(1 + zA)$. In order for this definition to make sense, the operator A is required to be *trace-class*, we refer to [39] for details. If so, the logarithm of the Fredholm determinant is given by

$$\begin{aligned} \log \det(1 + zA) &= \text{tr} \log(1 + zA) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1} z^m}{m} \int du_1 \dots du_m A(u_1, u_2) A(u_2, u_3) \dots A(u_m, u_1). \end{aligned} \quad (\text{A.3})$$

Another useful identity for Fredholm determinants is the cycle property. If A and B are Hilbert-Schmidt operators (*i.e.* $\text{tr } AA^* < \infty$ and $\text{tr } BB^* < \infty$), then

$$\det(1 + AB) = \det(1 + BA) . \quad (\text{A.4})$$

This property allows to exchange the roles of integrations which are inside the defining kernel of AB with the integration corresponding to the Fredholm determinant. We emphasize that the two kernels AB and BA do not necessarily act on the same space.

Numerical evaluations of Fredholm determinants can be performed by discretizing the integrals in (A.2). Using the von Koch formula (A.1), the evaluation of a Fredholm determinant is thereby reduced to the computation of the determinant of a finite matrix. We refer to [38, 40] for an illuminating discussion and precise error estimates.

Appendix B. Airy operator and Airy Hamiltonian

We recall the definition of the Airy Hamiltonian H and of the Airy operator K . We work in the space of complex-valued square integrable functions $L^2(\mathbb{R})$ with the scalar product

$$\langle f | g \rangle = \int_{-\infty}^{\infty} du f(u)^* g(u) . \quad (\text{B.1})$$

The Airy Hamiltonian H is defined by

$$H = -(\partial_u)^2 + u . \quad (\text{B.2})$$

If necessary, we write H as H_u to indicate the variable on which the Airy operator is acting. The Airy function is the solution of the differential equation

$$\text{Ai}''(u) = u \text{Ai}(u) \quad (\text{B.3})$$

such that $\text{Ai}(u) \rightarrow 0$ as $u \rightarrow \infty$. Setting

$$\phi_z(u) = \text{Ai}(u - z) , \quad (\text{B.4})$$

one notes that ϕ_z satisfies the eigenvalue equation

$$H\phi_z = z\phi_z . \quad (\text{B.5})$$

In addition,

$$\int_{-\infty}^{\infty} dz \text{Ai}(u - z) \text{Ai}(u' - z) = \delta(u - u') . \quad (\text{B.6})$$

In Dirac notation this completeness relation reads

$$\mathbb{1} = \int_{-\infty}^{\infty} dz |\phi_z\rangle \langle \phi_z| . \quad (\text{B.7})$$

Hence the Airy Hamiltonian has the spectral representation

$$H = \int_{-\infty}^{\infty} dz z |\phi_z\rangle \langle \phi_z| . \quad (\text{B.8})$$

The projection to all negative energy states defines the Airy operator

$$K = \int_{-\infty}^0 dz |\phi_z\rangle \langle \phi_z|. \quad (\text{B.9})$$

In particular, one has $K = K^*$, $K^2 = K$, and obviously $[K, H] = 0$. In position representation, the Airy kernel writes

$$\langle u|K|v\rangle = \int_0^\infty dz \text{Ai}(u+z) \text{Ai}(z+v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}'(u)\text{Ai}(v)}{u-v}. \quad (\text{B.10})$$

We also define the projection onto the spatial interval $[a, \infty)$ by

$$P_a = \int_a^\infty du |u\rangle \langle u|. \quad (\text{B.11})$$

The operator $P_a K P_a$ is trace class for all $a > -\infty$. $F_2(a) = \det(1 - P_a K P_a)$ is by definition the Tracy-Widom distribution [33] corresponding to the Gaussian Unitary Ensemble of random matrices.

Appendix C. Integration over q and summation over n in the kernel N

We start from the explicit expression

$$\begin{aligned} N(z, z') &= \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^\infty \frac{dq}{2\pi} \sum_{n=1}^\infty e^{iq(z-z')} e^{tn^3/24} e^{-x\partial_1\partial_2} \\ &\quad \times \left[-e^{-\frac{1}{2}(z+z')} e^{-tq^2/2} (e^{ix_1q-s_1} + e^{ix_2q-s_2}) \right]^n. \end{aligned} \quad (\text{C.1})$$

In order to perform the summation over n and the integration over q , we insert the classical relation

$$e^{tn^3/24} = \int_{-\infty}^\infty du \text{Ai}(u) e^{(t/8)^{1/3}nu}. \quad (\text{C.2})$$

Besides the Airy function there are infinitely many other functions which satisfy (C.2). One concrete example would be

$$\text{Ai}(u) + \sin(\pi u) e^{-u^2/2}. \quad (\text{C.3})$$

Our choice is determined by being the only one which provides the correct one-point result. Using the binomial theorem to expand the term of power n , one obtains

$$\begin{aligned} N(z, z') &= \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^\infty \frac{dq du}{2\pi} \sum_{n=1}^\infty \sum_{k=0}^n \text{Ai}(u) e^{iq[z-z'+x_1k+x_2(n-k)]} \\ &\quad \times \binom{n}{k} \left(-e^{(t/8)^{1/3}u} e^{-\frac{1}{2}(z+z')} e^{-tq^2/2} \right)^n e^{-x\partial_1\partial_2} e^{-s_1k-s_2(n-k)}. \end{aligned} \quad (\text{C.4})$$

We introduce the parameter $\alpha = (t/2)^{1/3}$ and perform the change of variable $u \rightarrow u + 2^{2/3}\alpha^2q^2 + 2^{-1/3}\alpha^{-1}(z+z')$ in the integral. This results in

$$\begin{aligned} N(z, z') &= \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^\infty \frac{dq du}{2\pi} \sum_{n=1}^\infty \sum_{k=0}^n \text{Ai}(u + 2^{2/3}\alpha^2q^2 + 2^{-1/3}\alpha^{-1}(z+z')) \\ &\quad \times e^{iq[z-z'+x_1k+x_2(n-k)]} \binom{n}{k} (-1)^n e^{2^{-2/3}\alpha un} e^{-x\partial_1\partial_2} e^{-s_1k-s_2(n-k)}. \end{aligned} \quad (\text{C.5})$$

Using the relation

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \text{Ai}(aq^2 + b) e^{icq} = 2^{-1/3} a^{-1/2} \text{Ai}(2^{-2/3}(b + a^{-1/2}c)) \text{Ai}(2^{-2/3}(b - a^{-1/2}c)) , \quad (\text{C.6})$$

see [41], the integration over q can be performed. One obtains

$$\begin{aligned} N(z, z') &= \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^n e^{2^{-2/3} \alpha u n} e^{-x \partial_1 \partial_2} e^{-s_1 k - s_2(n-k)} \\ &\quad \times 2^{-2/3} \alpha^{-1} \text{Ai}(2^{-2/3} u + \alpha^{-1} z + (2\alpha)^{-1} [x_1 k + x_2(n-k)]) \\ &\quad \times \text{Ai}(2^{-2/3} u + \alpha^{-1} z' - (2\alpha)^{-1} [x_1 k + x_2(n-k)]) . \end{aligned} \quad (\text{C.7})$$

We change variables as $u \rightarrow 2^{2/3} u$, $z \rightarrow \alpha z$, and use the relation

$$f(z + a) = \exp[a \partial_z] f(z) \quad (\text{C.8})$$

to move n and k out of the Airy functions. In preparation for the summation over n and k , one finds

$$\begin{aligned} \alpha N(\alpha z, \alpha z') &= \mathbb{1}_{\{z, z' > 0\}} \int_{-\infty}^{\infty} du \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^n e^{\alpha u n} e^{-x \partial_1 \partial_2} e^{-s_1 k - s_2(n-k)} \\ &\quad \times e^{(2\alpha)^{-1} (x_1 k + x_2(n-k)) (\partial_z - \partial_{z'})} \text{Ai}(u + z) \text{Ai}(u + z') . \end{aligned} \quad (\text{C.9})$$

Noticing the identity

$$e^{(2\alpha)^{-1} (x_1 k + x_2(n-k)) (\partial_z - \partial_{z'})} e^{-s_1 k - s_2(n-k)} = e^{-(2\alpha)^{-1} (x_2 \partial_2 + x_1 \partial_1) (\partial_z - \partial_{z'})} e^{-s_1 k - s_2(n-k)} , \quad (\text{C.10})$$

the summations over k and n finally yield the expression (4.18), (4.19) for the kernel N with Φ is defined as in Eq. (2.27).

References

- [1] T. Halpin-Healy and Y.-C. Zhang. Kinetic roughening phenomena, stochastic growth, directed polymers and all that. Aspects of multidisciplinary statistical mechanics. 1995 *Phys. Rep.*, 254:215–414.
- [2] M. Kardar. Statistical physics of fields. 2007 Cambridge University Press.
- [3] S. Lemerle, J. Ferré, C. Chappert, V. Mathet, T. Giamarchi, and P. Le Doussal. Domain wall creep in an Ising ultrathin magnetic film. 1998 *Phys. Rev. Lett.*, 80:849–852.
- [4] J. Kertész. Fractal fracture. 1992 *Physica A*, 191:208–212.
- [5] G. Blatter, M.V. Feigel'man, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur. Vortices in high-temperature superconductors. 1994 *Rev. Mod. Phys.*, 66:1125–1388.
- [6] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor. Nonperturbative renormalization group for the Kardar-Parisi-Zhang equation. 2010 *Phys. Rev. Lett.*, 104:150601.
- [7] H. Lacoïn. New bounds for the free energy of directed polymers in dimension 1+1 and 1+2. 2010 *Comm. Math. Phys.*, 294:471–503.
- [8] T. Sasamoto and H. Spohn. One-dimensional Kardar-Parisi-Zhang equation: an exact solution and its universality. 2010 *Phys. Rev. Lett.*, 104:230602.
- [9] T. Sasamoto and H. Spohn. Exact height distributions for the KPZ equation with narrow wedge initial condition. 2010 *Nuclear Phys. B*, 834:523–542.
- [10] G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions. 2010 *Comm. Pure Appl. Math.*, online.

- [11] T. Sasamoto and H. Spohn. The crossover regime for the weakly asymmetric simple exclusion process. 2010 *J. Stat. Phys.*, 140:209–231.
- [12] P. Calabrese, P. Le Doussal, and A. Rosso. Free-energy distribution of the directed polymer at high temperature. 2010 *EPL*, 90:20002.
- [13] V. Dotsenko. Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers. 2010 *EPL*, 90:20003.
- [14] V. Dotsenko. Replica Bethe ansatz derivation of the Tracy-Widom distribution of the free energy fluctuations in one-dimensional directed polymers. 2010 *J. Stat. Mech.*, P07010.
- [15] M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamic scaling of growing interfaces. 1986 *Phys. Rev. Lett.*, 56:889–892.
- [16] T. Sasamoto and H. Spohn. The 1+1-dimensional Kardar-Parisi-Zhang equation and its universality class. 2010 *J. Stat. Mech.*, P11013.
- [17] E.H. Lieb and W. Liniger. Exact analysis of an interacting Bose gas. I. the general solution and the ground state. 1963 *Phys. Rev.*, 130:1605–1616.
- [18] J.B. McGuire. Study of exactly soluble one-dimensional N-body problems. 1964 *J. Math. Phys.*, 5:622–636.
- [19] M. Kardar. Replica Bethe ansatz studies of two-dimensional interfaces with quenched random impurities. 1987 *Nucl. Phys. B*, 290:582–602.
- [20] J.P. Bouchaud and H. Orland. On the Bethe ansatz for random directed polymers. 1990 *J. Stat. Phys.*, 61:877–884.
- [21] M. Prähofer and H. Spohn. Scale invariance of the PNG droplet and the Airy process. 2002 *J. Stat. Phys.*, 108:1071–1106.
- [22] V. Dotsenko and B. Klumov. Bethe ansatz solution for one-dimensional directed polymers in random media. 2010 *J. Stat. Mech.*, P03022.
- [23] E. Brunet and B. Derrida. Probability distribution of the free energy of a directed polymer in a random medium. 2000 *Phys. Rev. E*, 61:6789–6801.
- [24] B. Derrida and J.L. Lebowitz. Exact large deviation function in the asymmetric exclusion process. 1998 *Phys. Rev. Lett.*, 80:209–213.
- [25] D.S. Lee and D. Kim. Large deviation function of the partially asymmetric exclusion process. 1999 *Phys. Rev. E*, 59:6476–6482.
- [26] S. Prolhac and K. Mallick. Cumulants of the current in a weakly asymmetric exclusion process. 2009 *J. Phys. A: Math. Theor.*, 42:175001.
- [27] S. Prolhac. Tree structures for the current fluctuations in the exclusion process. 2010 *J. Phys. A: Math. Theor.*, 43:105002.
- [28] L.-H. Gwa and H. Spohn. Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian. 1992 *Phys. Rev. Lett.*, 68:725–728.
- [29] D. Kim. Bethe Ansatz solution for crossover scaling functions of the asymmetric XXZ chain and the Kardar-Parisi-Zhang-type growth model. 1995 *Phys. Rev. E*, 52:3512–3524.
- [30] O. Golinelli and K. Mallick. Spectral gap of the totally asymmetric exclusion process at arbitrary filling. 2005 *J. Phys. A: Math. Gen.*, 38:1419–1425.
- [31] O. Golinelli and K. Mallick. Spectral degeneracies in the totally asymmetric exclusion process. 2005 *J. Stat. Phys.*, 120:779–798.
- [32] C.A. Tracy and H. Widom. Integral formulas for the asymmetric simple exclusion process. 2008 *Commun. Math. Phys.*, 279:815–844.
- [33] C.A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. 1994 *Commun. Math. Phys.*, 159:151–174.
- [34] N. O’Connell. Directed polymers and the quantum Toda lattice. arXiv:0910.0069v4.
- [35] K. Johansson. Shape fluctuations and random matrices. 2000 *Commun. Math. Phys.*, 209:437–476.
- [36] K. Johansson. The arctic circle boundary and the Airy process. 2005 *Annals Prob.*, 33:1–30.
- [37] K.A. Takeuchi and M. Sano. Universal fluctuations of growing interfaces: evidence in turbulent

- liquid crystals. 2010 *Phys. Rev. Lett.*, 104:230601.
- [38] F. Bornemann. On the numerical evaluation of Fredholm determinants. 2010 *Math. Comp.*, 79:871–915.
- [39] B. Simon. Trace Ideals and Their Applications, volume 120 of *Mathematical Surveys and Monographs*. 2005 Amer. Math. Soc., second edition.
- [40] F. Bornemann. On the numerical evaluation of distributions in Random Matrix Theory: A review. arXiv:0904.1581.
- [41] O. Vallée, M. Soares, and C. de Izarra. An integral representation for the product of Airy functions. 1997 *Z. Angew. Math. Phys.*, 48:156–160.